

# Stability for $t$ -intersecting families of permutations

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## Abstract

A family of permutations  $\mathcal{A} \subset S_n$  is said to be  $t$ -intersecting if any two permutations in  $\mathcal{A}$  agree on at least  $t$  points, i.e. for any  $\sigma, \pi \in \mathcal{A}$ ,  $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$ . It was proved by Friedgut, Pilpel and the author in [6] that for  $n$  sufficiently large depending on  $t$ , a  $t$ -intersecting family  $\mathcal{A} \subset S_n$  has size at most  $(n-t)!$ , with equality only if  $\mathcal{A}$  is a coset of the stabilizer of  $t$  points (or ‘ $t$ -coset’ for short), proving a conjecture of Deza and Frankl. Here, we first obtain a rough stability result for  $t$ -intersecting families of permutations, namely that for any  $t \in \mathbb{N}$  and any positive constant  $c$ , if  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family of permutations of size at least  $c(n-t)!$ , then there exists a  $t$ -coset containing all but at most a  $O(1/n)$ -fraction of  $\mathcal{A}$ . We use this to prove an exact stability result: for  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family which is not contained within a  $t$ -coset, then  $\mathcal{A}$  is at most as large as the family

$$\begin{aligned} \mathcal{D} = & \{ \sigma \in S_n : \sigma(i) = i \ \forall i \leq t, \sigma(j) = j \text{ for some } j > t+1 \} \\ & \cup \{ (1 \ t+1), (2 \ t+1), \dots, (t \ t+1) \} \end{aligned}$$

which has size  $(1 - 1/e + o(1))(n-t)!$ . Moreover, if  $\mathcal{A}$  is the same size as  $\mathcal{D}$  then it must be a ‘double translate’ of  $\mathcal{D}$ , meaning that there exist  $\pi, \tau \in S_n$  such that  $\mathcal{A} = \pi\mathcal{D}\tau$ . The  $t = 1$  case of this was a conjecture of Cameron and Ku and was proved by the author in [5]. We build on the methods of [5], but the representation theory of  $S_n$  and the combinatorial arguments are more involved. We also obtain an analogous result for  $t$ -intersecting families in the alternating group  $A_n$ .

## 1 Introduction

We work first on the symmetric group  $S_n$ , the group of all permutations of  $\{1, 2, \dots, n\} = [n]$ . A family of permutations  $\mathcal{A} \subset S_n$  is said to be  $t$ -intersecting if any two permutations in  $\mathcal{A}$  agree on at least  $t$  points, i.e. for

any  $\sigma, \pi \in \mathcal{A}$ ,  $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$ . Deza and Frankl [4] conjectured that for  $n$  sufficiently large depending on  $t$ , a  $t$ -intersecting family  $\mathcal{A} \subset S_n$  has size at most  $(n-t)!$ ; this became known as the Deza-Frankl conjecture. It was proved in 2008 by Friedgut, Pilpel and the author in [6] using eigenvalue techniques and the representation theory of the symmetric group; it was also shown in [6] that equality holds only if  $\mathcal{A}$  is a coset of the stabilizer of  $t$  points (or ‘ $t$ -coset’ for short). In this paper, we will first prove a rough stability result for  $t$ -intersecting families of permutations. Namely, we show that for any fixed  $t \in \mathbb{N}$  and  $c > 0$ , if  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family of size at least  $c(n-t)!$ , then there exists a  $t$ -coset  $\mathcal{C}$  such that  $|\mathcal{A} \setminus \mathcal{C}| \leq \Theta((n-t-1)!)$ , i.e.  $\mathcal{C}$  contains all but at most a  $O(1/n)$ -fraction of  $\mathcal{A}$ .

We then use some additional combinatorial arguments to prove an exact stability result: for  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family which is not contained within a  $t$ -coset, then  $\mathcal{A}$  is at most as large as the family

$$\begin{aligned} \mathcal{D} = & \{ \sigma \in S_n : \sigma(i) = i \ \forall i \leq t, \sigma(j) = j \text{ for some } j > t+1 \} \\ & \cup \{(1 \ t+1), (2 \ t+1), \dots, (t \ t+1)\} \end{aligned}$$

which has size  $(1 - 1/e + o(1))(n-t)!$ . Moreover, if  $\mathcal{A}$  is the same size as  $\mathcal{D}$ , then it must be a ‘double translate’ of  $\mathcal{D}$ , meaning that there exist  $\pi, \tau \in S_n$  such that  $\mathcal{A} = \pi \mathcal{D} \tau$ . Note that if  $\mathcal{F} \subset S_n$ , any double translate of  $\mathcal{F}$  has the same size as  $\mathcal{F}$ , is  $t$ -intersecting iff  $\mathcal{F}$  is and is contained within a  $t$ -coset of  $S_n$  iff  $\mathcal{F}$  is; this will be our notion of ‘isomorphism’.

In other words, if we demand that our  $t$ -intersecting family  $\mathcal{A} \subset S_n$  is not contained within a  $t$ -coset of  $S_n$ , then it is best to take  $\mathcal{A}$  such that all but  $t$  of its permutations are contained within some  $t$ -coset.

One may compare this with the situation for  $t$ -intersecting families of  $r$ -sets. We say a family  $\mathcal{A} \subset [n]^{(r)}$  of  $r$ -element subsets of  $[n]$  is  *$t$ -intersecting* if any two of its sets contain at least  $t$  elements in common, i.e.  $|x \cap y| \geq t$  for any  $x, y \in \mathcal{A}$ . Wilson [11] proved using an eigenvalue technique that provided  $n \geq (t+1)(r-t+1)$ , a  $t$ -intersecting family  $\mathcal{A} \subset [n]^{(r)}$  has size at most  $\binom{n-t}{r-t}$ , and that for  $n > (t+1)(r-t+1)$ , equality holds only if  $\mathcal{A}$  consists of all  $r$ -sets containing some fixed  $t$ -set. Later, Ahlswede and Khachatrian [1] characterized the  $t$ -intersecting families of maximum size in  $[n]^{(r)}$  for all values of  $t, r$  and  $n$  using entirely combinatorial methods based on left-compression. They also proved that for  $n > (t+1)(r-t+1)$ , if  $\mathcal{A} \subset [n]^{(r)}$  is  $t$ -intersecting and *non-trivial*, meaning that there is no  $t$ -set contained in all of its members, then  $\mathcal{A}$  is at most as large as the family

$$\{x \in [n]^{(r)} : [t] \subset x, x \cap \{t+1, \dots, r+1\} \neq \emptyset\} \cup \{[r+1] \setminus \{i\} : i \in [t]\}$$

if  $r > 2t + 1$ , and at most as large as the family

$$\{x \in [n]^{(r)} : |x \cap [t+2]| \geq t+1\}$$

if  $r \leq 2t + 1$ . This had been proved under the assumption  $n \geq n_1(r, t)$  by Frankl [7] in 1978. Note that the first family above is ‘almost trivial’, and is the natural analogue of our family  $\mathcal{D}$ .

The  $t = 1$  case of our result was a conjecture of Cameron and Ku and was proved by the author in [5]. We build on the methods of [5], but the representation theory of  $S_n$  and the combinatorial arguments required are more involved.

We also obtain analogous results for  $t$ -intersecting families of permutations in the alternating group  $A_n$ . We use the methods of [6] to show that for  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A} \subset A_n$  is  $t$ -intersecting, then  $|\mathcal{A}| \leq (n-t)!/2$ . Interestingly, it does not seem possible to use the methods of [6] to show that equality holds only if  $\mathcal{A}$  is a coset of the stabilizer of  $t$  points. Instead, we deduce this from a stability result. Using the same techniques as for  $S_n$ , we prove that if  $\mathcal{A} \subset A_n$  is  $t$ -intersecting but not contained within a  $t$ -coset, then it is at most as large as the family

$$\begin{aligned} \mathcal{E} = & \{ \sigma \in A_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = (n-1 \ n)(j) \text{ for some } j > t+1 \} \\ & \cup \{ (1 \ t+1)(n-1 \ n), (2 \ t+1)(n-1 \ n), \dots, (t \ t+1)(n-1 \ n) \} \end{aligned}$$

which has size  $(1 - 1/e + o(1))(n-t)!/2$ ; if  $\mathcal{A}$  is the same size as  $\mathcal{E}$ , then it must be a double translate of  $\mathcal{E}$ , meaning that  $\mathcal{A} = \pi\mathcal{E}\tau$  for some  $\pi, \tau \in A_n$ .

## 2 Background

In [6], in order to prove the Deza-Frankl conjecture, we constructed (for  $n$  sufficiently large depending on  $t$ ) a weighted graph  $Y$  which was a real linear combination of Cayley graphs on  $S_n$  generated by conjugacy-classes of permutations with less than  $t$  fixed points, such that the matrix  $A$  of weights of  $Y$  had maximum eigenvalue 1 and minimum eigenvalue

$$\omega_{n,t} = -\frac{1}{n(n-1)\dots(n-t+1)-1}$$

The 1-eigenspace was the subspace of  $\mathbb{C}[S_n]$  consisting of the constant functions. The direct sum of the 1-eigenspace and the  $\omega_{n,t}$ -eigenspace was the subspace  $V_t$  of  $\mathbb{C}[S_n]$  spanned by the characteristic vectors of the  $t$ -cosets of  $S_n$ . All other eigenvalues were  $O(|\omega_{n,t}|/n^{1/6})$ ; this can in fact be improved to

$O(|\omega_{n,t}|/n)$ , but any bound of the form  $o(|\omega_{n,t}|)$  will suffice for our purposes. We then appealed to a weighted version of Hoffman's bound (Theorem 11 in [6]):

**Theorem 1.** *Let  $A$  be a real, symmetric,  $N \times N$  matrix with real eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$  (where  $\lambda_1 > 0$ ), such that the all-1's vector  $\mathbf{f}$  is an eigenvector of  $A$  with eigenvalue  $\lambda_1$ , i.e. all row and column sums of  $A$  equal  $\lambda_1$ . Let  $X \subset [N]$  such that  $A_{x,y} = 0$  for any  $x, y \in X$ . Let  $U$  be the direct sum of the subspace of constant vectors and the  $\lambda_N$ -eigenspace. Then*

$$|X| \leq \frac{|\lambda_N|}{\lambda_1 + |\lambda_N|} N$$

and equality holds only if the characteristic vector  $v_X$  lies in the subspace  $U$ .

Applying this to our weighted graph  $Y$  proved the Deza-Frankl conjecture:

**Theorem 2.** *For  $n$  sufficiently large depending on  $t$ , a  $t$ -intersecting family  $\mathcal{A} \subset S_n$  has size  $|\mathcal{A}| \leq (n-t)!$ .*

Note that equality holds only if the characteristic vector  $v_{\mathcal{A}}$  of  $\mathcal{A}$  lies in the subspace  $V_t$  spanned by the characteristic vectors of the  $t$ -cosets of  $S_n$ . It was proved in [6] that the Boolean functions in  $V_t$  are precisely the disjoint unions of  $t$ -cosets of  $S_n$ , implying that equality holds only if  $\mathcal{A}$  is a  $t$ -coset of  $S_n$ .

We also appealed to the following cross-independent weighted version of Hoffman's bound:

**Theorem 3.** *Let  $A$  be as in Theorem 1, and let  $\nu = \max(|\lambda_2|, |\lambda_N|)$ . Let  $X, Y \subset [N]$  such that  $A_{x,y} = 0$  for any  $x \in X$  and  $y \in Y$ . Let  $U$  be the direct sum of the subspace of constant vectors and the  $\pm\nu$ -eigenspaces. Then*

$$|X||Y| \leq \left( \frac{\nu}{\lambda_1 + \nu} N \right)^2$$

and equality holds only if  $|X| = |Y|$  and the characteristic vectors  $v_X$  and  $v_Y$  lie in the subspace  $U$ .

Applying this to our weighted graph  $Y$  yielded:

**Theorem 4.** *For  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A}, \mathcal{B} \subset S_n$  are  $t$ -cross-intersecting, then  $|\mathcal{A}||\mathcal{B}| \leq ((n-t)!)^2$ .*

This will be a crucial tool in our stability analysis. Note that if equality holds in Theorem 4, then the characteristic vectors  $v_{\mathcal{A}}$  and  $v_{\mathcal{B}}$  lie in the subspace  $V_t$  spanned by the characteristic vectors of the  $t$ -cosets of  $S_n$ , so by the same argument as before,  $\mathcal{A}$  and  $\mathcal{B}$  must both be equal to the same  $t$ -coset of  $S_n$ .

We will need the following ‘stability’ version of Theorem 1:

**Lemma 5.** *Let  $A$ ,  $X$  and  $U$  be as in Theorem 1. Let  $\alpha = |X|/N$ . Let  $\lambda_M$  be the negative eigenvalue of second largest modulus. Equip  $\mathbb{C}^N$  with the inner product:*

$$\langle x, y \rangle = \frac{1}{N} \sum_{i=1}^N \bar{x}_i y_i$$

and let

$$||x|| = \sqrt{\frac{1}{N} \sum_{i=1}^N |x_i|^2}$$

be the induced norm. Let  $D$  be the Euclidean distance from the characteristic vector  $v_X$  of  $X$  to the subspace  $U$ , i.e. the norm  $||P_{U^\perp}(v_X)||$  of the projection of  $v_X$  onto  $U^\perp$ . Then

$$D^2 \leq \frac{(1 - \alpha)|\lambda_N| - \lambda_1 \alpha}{|\lambda_N| - |\lambda_M|} \alpha$$

For completeness, we include a proof:

*Proof.* Let  $u_1 = \mathbf{f}, u_2, \dots, u_N$  be an orthonormal basis of real eigenvectors of  $A$  corresponding to the eigenvalues  $\lambda_1, \dots, \lambda_N$ . Write

$$v_X = \sum_{i=1}^N \xi_i u_i$$

as a linear combination of the eigenvectors of  $A$ ; we have  $\xi_1 = \alpha$  and

$$\sum_{i=1}^N \xi_i^2 = ||v_X||^2 = |X|/N = \alpha$$

Then we have the crucial property:

$$0 = \sum_{x,y \in X} A_{x,y} = v_X^\top A v_X = \sum_{i=1}^N \lambda_i \xi_i^2 \geq \lambda_1 \xi_1^2 + \lambda_N \sum_{i: \lambda_i = \lambda_N} \xi_i^2 + \lambda_M \sum_{i > 1: \lambda_i \neq \lambda_N} \xi_i^2$$

Note that

$$\sum_{i>1:\lambda_i\neq\lambda_N} \xi_i^2 = D^2$$

and

$$\sum_{i:\lambda_i=\lambda_N} \xi_i^2 = \alpha - \alpha^2 - D^2$$

so we have

$$0 \geq \lambda_1 \alpha^2 + \lambda_N (\alpha - \alpha^2 - D^2) + \lambda_M D^2$$

Rearranging, we obtain:

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - \lambda_1 \alpha}{|\lambda_N| - |\lambda_M|} \alpha$$

as required.  $\square$

Our weighted graph  $Y$  has  $\lambda_N = \omega_{n,t}$  and  $|\lambda_M| = O(|\omega_{n,t}|/n^{1/6})$ , so applying the above result to a  $t$ -intersecting family  $\mathcal{A} \subset S_n$  gives:

$$\|P_{V_t^\perp}(v_{\mathcal{A}})\|^2 \leq (1 - |\mathcal{A}|/(n-t)!)(1 + O(n^{1/6}))|\mathcal{A}|/n! \quad (1)$$

Next, we find a formula for the projection  $P_{V_t}(v_{\mathcal{A}})$  of the characteristic vector of  $\mathcal{A}$  onto the subspace  $V_t$  spanned by the characteristic vectors of the  $t$ -cosets of  $S_n$ . But first, we need some background on non-Abelian Fourier analysis and the representation theory of the symmetric group.

## Background from non-Abelian Fourier analysis

We now recall some information we need from [6]. [Notes for algebraists are included in square brackets and may be ignored without prejudicing the reader's understanding.]

If  $G$  is a finite group, a *representation* of  $G$  is a vector space  $W$  together with a group homomorphism  $\rho : G \rightarrow \text{GL}(W)$  from  $G$  to the group of all automorphisms of  $W$ , or equivalently a linear action of  $G$  on  $W$ . If  $W = \mathbb{C}^m$ , then  $\text{GL}(W)$  can be identified with the group of all complex invertible  $m \times m$  matrices; we call  $\rho$  a *complex matrix representation* of degree (or dimension)  $m$ . [Note that  $\rho$  makes  $\mathbb{C}^m$  into a  $\mathbb{C}G$ -module of dimension  $m$ .]

We say a representation  $(\rho, W)$  is *irreducible* if it has no proper subrepresentation, i.e. no proper subspace of  $W$  is fixed by  $\rho(g)$  for every  $g \in G$ . We say that two (complex) representations  $(\rho, W)$  and  $(\rho', W')$  are *equivalent* if there exists a linear isomorphism  $\phi : W \rightarrow W'$  such that  $\rho'(g) \circ \phi = \phi \circ \rho(g) \forall g \in G$ .

For any finite group  $G$ , there are only finitely many equivalence classes of irreducible complex representations of  $G$ . Let  $(\rho_1, \rho_2, \dots, \rho_k)$  be a complete set of pairwise non-equivalent complex irreducible matrix representations of  $G$  (i.e. containing one from each equivalence class of complex irreducible representations).

**Definition 1.** *The (non-Abelian) Fourier transform of a function  $f : G \rightarrow \mathbb{C}$  at the irreducible representation  $\rho_i$  is the matrix*

$$\hat{f}(\rho_i) = \frac{1}{|G|} \sum_{g \in G} f(g) \rho_i(g)$$

Let  $V_{\rho_i}$  be the subspace of functions whose Fourier transform is concentrated on  $\rho_i$ , i.e. with  $\hat{f}(\rho_j) = 0$  for each  $j \neq i$ . [Identifying the space  $\mathbb{C}[G]$  of all complex-valued functions on  $G$  with the *group module*  $\mathbb{C}G$ ,  $V_{\rho_i}$  is the sum of all submodules of the group module isomorphic to the module defined by  $\rho_i$ ; it has dimension  $\dim(V_{\rho_i}) = (\dim(\rho_i))^2$ . The group module decomposes as

$$\mathbb{C}G = \bigoplus_{i=1}^k V_{\rho_i}$$

Write  $\text{Id} = \sum_{i=1}^k e_i$ , where  $e_i \in V_{\rho_i}$  for each  $i \in [k]$ . The  $e_i$ 's are called the *primitive central idempotents* of  $\mathbb{C}G$ ; they are given by the following formula:

$$e_i = \frac{\dim(\rho_i)}{|G|} \sum_{g \in G} \chi_i(g^{-1})g$$

They are in the *centre*  $Z(\mathbb{C}G)$  of the group module, and satisfy  $e_i e_j = \delta_{i,j}$ . Note that  $V_{\rho_i}$  is the two-sided ideal of  $\mathbb{C}G$  generated by  $e_i$ . For any  $x \in \mathbb{C}G$ , the unique decomposition of  $x$  into elements of the  $V_{\rho_i}$ 's is given by  $x = \sum_{i=1}^k e_i x$ .

A function  $f : G \rightarrow \mathbb{C}$  may be recovered from its Fourier transform using the Fourier Inversion Formula:

$$f(g) = \sum_{i=1}^k \dim(\rho_i) \text{Tr} \left( \hat{f}(\rho_i) \rho_i(g^{-1}) \right)$$

where  $\text{Tr}(M)$  denotes the trace of the matrix  $M$ . It follows from this that the projection of  $f$  onto  $V_{\rho_i}$  has  $g$ -coordinate

$$P_{V_{\rho_i}}(f)_g = \frac{\dim(\rho_i)}{|G|} \sum_{h \in G} f(h) \text{Tr}(\rho_i(hg^{-1})) = \frac{\dim(\rho_i)}{|G|} \sum_{h \in G} f(h) \chi_{\rho_i}(hg^{-1})$$

where  $\chi_{\rho_i}(g) = \text{Tr}(\rho_i(g))$  denotes the character of the representation  $\rho_i$ .

## Background on the representation theory of $S_n$

A *partition* of  $n$  is a non-increasing sequence of positive integers summing to  $n$ , i.e. a sequence  $\alpha = (\alpha_1, \dots, \alpha_k)$  with  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_l \geq 1$  and  $\sum_{i=1}^l \alpha_i = n$ ; we write  $\alpha \vdash n$ . For example,  $(3, 2, 2) \vdash 7$ ; we sometimes use the shorthand  $(3, 2, 2) = (3, 2^2)$ .

The *cycle-type* of a permutation  $\sigma \in S_n$  is the partition of  $n$  obtained by expressing  $\sigma$  as a product of disjoint cycles and listing its cycle-lengths in non-increasing order. The conjugacy-classes of  $S_n$  are precisely

$$\{\sigma \in S_n : \text{cycle-type}(\sigma) = \alpha\}_{\alpha \vdash n}.$$

Moreover, there is an explicit 1-1 correspondence between irreducible representations of  $S_n$  (up to isomorphism) and partitions of  $n$ , which we now describe.

Let  $\alpha = (\alpha_1, \dots, \alpha_l)$  be a partition of  $n$ . The *Young diagram* of  $\alpha$  is an array of  $n$  dots, or cells, having  $l$  left-justified rows where row  $i$  contains  $\alpha_i$  dots. For example, the Young diagram of the partition  $(3, 2^2)$  is

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If the array contains the numbers  $\{1, 2, \dots, n\}$  in some order in place of the dots, we call it an  $\alpha$ -*tableau*; for example,

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6  1  7
5  4
3  2

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is a  $(3, 2^2)$ -tableau. Two  $\alpha$ -tableaux are said to be *row-equivalent* if for each row, they have the same numbers in that row. If an  $\alpha$ -tableau  $s$  has rows  $R_1, \dots, R_l \subset [n]$  and columns  $C_1, \dots, C_k \subset [n]$ , we let  $R_s = S_{R_1} \times S_{R_2} \times \dots \times S_{R_l}$  be the row-stabilizer of  $s$  and  $C_s = S_{C_1} \times S_{C_2} \times \dots \times S_{C_k}$  be the column-stabilizer.

An  $\alpha$ -*tabloid* is an  $\alpha$ -tableau with unordered row entries (or formally, a row-equivalence class of  $\alpha$ -tableaux); given a tableau  $s$ , we write  $[s]$  for the tabloid it produces. For example, the  $(3, 2^2)$ -tableau above produces the following  $(3, 2^2)$ -tabloid



$$\begin{array}{ccc} \{1 & 6 & 7\} \\ \{4 & 5\} \\ \{2 & 3\} \end{array}$$

Consider the natural left action of  $S_n$  on the set  $X^\alpha$  of all  $\alpha$ -tabloids; let  $M^\alpha = \mathbb{C}[X^\alpha]$  be the corresponding permutation module, i.e. the complex vector space with basis  $X^\alpha$  and  $S_n$  action given by extending this action linearly. Given an  $\alpha$ -tableau  $s$ , we define the corresponding  $\alpha$ -polytabloid

$$e_s := \sum_{\pi \in C_s} \epsilon(\pi) \pi[s]$$

We define the *Specht module*  $S^\alpha$  to be the submodule of  $M^\alpha$  spanned by the  $\alpha$ -polytabloids:

$$S^\alpha = \text{Span}\{e_s : s \text{ is an } \alpha\text{-tableau}\}.$$

A central observation in the representation theory of  $S_n$  is that *the Specht modules are a complete set of pairwise non-isomorphic, irreducible representations of  $S_n$* . Hence, any irreducible representation  $\rho$  of  $S_n$  is isomorphic to some  $S^\alpha$ . For example,  $S^{(n)} = M^{(n)}$  is the trivial representation;  $M^{(1^n)}$  is the left-regular representation, and  $S^{(1^n)}$  is the sign representation  $S$ .

We say that a tableau is *standard* if the numbers strictly increase along each row and down each column. It turns out that for any partition  $\alpha$  of  $n$ ,

$$\{e_t : t \text{ is a standard } \alpha\text{-tableau}\}$$

is a basis for the Specht module  $S^\alpha$ .

Given a partition  $\alpha$  of  $n$ , for each cell  $(i, j)$  in its Young diagram, we define the ‘hook-length’ ( $h_{i,j}^\alpha$ ) to be the number of cells in its ‘hook’ (the set of cells in the same row to the right of it or in the same column below it, including itself) — for example, the hook-lengths of  $(3, 2^2)$  are as follows:

$$\begin{array}{ccc} 5 & 4 & 1 \\ 3 & 2 & \\ 2 & 1 & \end{array}$$

The dimension  $f^\alpha$  of the Specht module  $S^\alpha$  is given by the following formula

$$f^\alpha = n! / \prod (\text{hook lengths of } [\alpha]) \quad (2)$$

From now on we will write  $[\alpha]$  for the equivalence class of the irreducible representation  $S^\alpha$ ,  $\chi_\alpha$  for the irreducible character  $\chi_{S^\alpha}$ , and  $\xi_\alpha$  for

the character of the permutation representation  $M^\alpha$ . Notice that the set of  $\alpha$ -tabloids form a basis for  $M^\alpha$ , and therefore  $\xi_\alpha(\sigma)$ , the trace of the corresponding permutation representation at  $\sigma$ , is precisely the number of  $\alpha$ -tabloids fixed by  $\sigma$ .

We now explain how the permutation modules  $M^\beta$  decompose into irreducibles.

**Definition 2.** Let  $\alpha, \beta$  be partitions of  $n$ . A generalized  $\alpha$ -tableau is produced by replacing each dot in the Young diagram of  $\alpha$  with a number between 1 and  $n$ ; if a generalized  $\alpha$ -tableau has  $\beta_i$   $i$ 's ( $1 \leq i \leq n$ ) it is said to have content  $\beta$ . A generalized  $\alpha$ -tableau is said to be semistandard if the numbers are non-decreasing along each row and strictly increasing down each column.

**Definition 3.** Let  $\alpha, \beta$  be partitions of  $n$ . The Kostka number  $K_{\alpha, \beta}$  is the number of semistandard generalized  $\alpha$ -tableaux with content  $\beta$ .

Young's Rule states that for any partition  $\beta$  of  $n$ , the permutation module  $M^\beta$  decomposes into irreducibles as follows:

$$M^\beta \cong \bigoplus_{\alpha \vdash n} K_{\alpha, \beta} S^\alpha$$

For example,  $M^{(n-1, 1)}$ , which corresponds to the natural permutation action of  $S_n$  on  $[n]$ , decomposes as

$$M^{(n-1, 1)} \cong S^{(n-1, 1)} \oplus S^{(n)}$$

and therefore

$$\xi_{(n-1, 1)} = \chi_{(n-1, 1)} + 1$$

Let  $V_\alpha$  be the subspace of  $\mathbb{C}[S_n]$  consisting of functions whose Fourier transform is concentrated on  $[\alpha]$ ; equivalently,  $V_\alpha$  is the sum of all submodules of  $\mathbb{C}S_n$  isomorphic to the Specht module  $S^\alpha$ .

We call a partition of  $n$  (or an irreducible representation of  $S_n$ ) 'fat' if its Young diagram has first row of length at least  $n - t$ . Let  $\mathcal{F}_{n, t}$  denote the set of all fat partitions of  $n$ ; note that for  $n \geq 2t$ ,

$$|\mathcal{F}_{n, t}| = \sum_{s=0}^t p(s)$$

where  $p(s)$  denotes the number of partitions of  $s$ . This grows very rapidly with  $t$ , but (as will be crucial for our stability analysis) it is independent of  $n$  for  $n \geq 2t$ . Note that  $\{[\alpha] : \alpha \text{ is fat}\}$  are precisely the irreducible constituents of the permutation module  $M^{(n-t, 1^t)}$  corresponding to the action

of  $S_n$  on  $t$ -tuples of distinct numbers, since  $K_{\alpha, (n-t, 1^t)} \geq 1$  iff there exists a semistandard generalized  $\alpha$ -tableau of content  $(n-t, 1^t)$ , i.e. iff  $\alpha_1 \geq n-t$ .

Recall from [6] that  $V_t$  is the subspace of functions whose Fourier transform is concentrated on the ‘fat’ irreducible representations of  $S_n$ ; equivalently,

$$V_t = \bigoplus_{\text{fat } \alpha} V_\alpha \quad (3)$$

The projection of  $u \in \mathbb{C}[S_n]$  onto  $V_\alpha$  has  $\sigma$ -coordinate

$$P_{V_\alpha}(u)_\sigma = \frac{f^\alpha}{n!} \sum_{\pi \in S_n} u(\pi) \chi_\alpha(\pi \sigma^{-1})$$

and therefore the projection of  $u$  onto  $V_t$  has  $\sigma$ -coordinate

$$P_{V_t}(u)_\sigma = \frac{1}{n!} \sum_{\text{fat } \alpha} f^\alpha \sum_{\pi \in S_n} u(\pi) \chi_\alpha(\pi \sigma^{-1}) \quad (4)$$

### 3 Stability

We are now in a position to prove our rough stability result:

**Theorem 6.** *Let  $t \in \mathbb{N}, c > 0$  be fixed. If  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family with  $|\mathcal{A}| \geq c(n-t)!$ , then there exists a  $t$ -coset  $\mathcal{C}$  such that  $|\mathcal{A} \setminus \mathcal{C}| \leq O((n-t-1)!)$ .*

In other words, if  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family of size at least a constant proportion of the maximum possible size  $(n-t)!$ , then there is some  $t$ -coset containing all but at most a  $O(1/n)$ -fraction of  $\mathcal{A}$ .

To prove this, we will first prove the following weaker statement:

**Lemma 7.** *Let  $t \in \mathbb{N}, c > 0$  be fixed. If  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family of size at least  $c(n-t)!$ , then there exist  $i$  and  $j$  such that all but at most  $O((n-t-1)!)$  permutations in  $\mathcal{A}$  map  $i$  to  $j$ .*

In other words, a large  $t$ -intersecting family is almost contained within a 1-coset. Theorem 6 will follow easily from this by an inductive argument.

Given distinct  $i_1, \dots, i_l$  and distinct  $j_1, \dots, j_l$ , we will write

$$\mathcal{A}_{i_1 \mapsto j_1, i_2 \mapsto j_2, \dots, i_l \mapsto j_l} := \{\sigma \in \mathcal{A} : \sigma(i_k) = j_k \ \forall k \in [l]\}$$

To prove Lemma 7, we will first observe from (1) that if  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family of size at least  $c(n-t)!$  then the characteristic vector

$v_{\mathcal{A}}$  of  $\mathcal{A}$  is close to the subspace  $V_t$  spanned by the characteristic vectors of the  $t$ -cosets. We will use this, combined with representation-theoretic arguments, to show that there exists some  $t$ -coset  $\mathcal{C}_0$  such that

$$|\mathcal{A} \cap \mathcal{C}_0| \geq \omega((n-2t)!)$$

—without loss of generality,  $\mathcal{C}_0 = \{\sigma \in S_n : \sigma(1) = 1, \dots, \sigma(t) = t\}$ , so

$$|\mathcal{A}_{1 \mapsto 1, 2 \mapsto 2, \dots, t \mapsto t}| \geq \omega((n-2t)!)$$

Note that the average size of the intersection of  $\mathcal{A}$  with a  $t$ -coset is

$$|\mathcal{A}|/n(n-1)\dots(n-t+1) = \Theta((n-2t)!)$$

We only know that  $\mathcal{A} \cap \mathcal{C}_0$  has size  $\omega$  of the average size. This statement would at first seem to weak to help us. However, for any distinct  $j_1 \neq 1, j_2 \neq 2, \dots$ , and  $j_t \neq t$ , the pair of families

$$\mathcal{A}_{1 \mapsto 1, 2 \mapsto 2, \dots, t \mapsto t}, \quad \mathcal{A}_{1 \mapsto j_1, 2 \mapsto j_2, \dots, t \mapsto j_t}$$

is  $t$ -cross-intersecting, so we may compare their sizes. In detail, we will deduce from Theorem 4 that

$$|\mathcal{A}_{1 \mapsto 1, 2 \mapsto 2, \dots, t \mapsto t}| |\mathcal{A}_{1 \mapsto j_1, 2 \mapsto j_2, \dots, t \mapsto j_t}| \leq ((n-2t)!)^2$$

giving  $|\mathcal{A}_{1 \mapsto j_1, \dots, t \mapsto j_t}| \leq o((n-2t)!)$ . Summing over all choices of  $j_1, \dots, j_t$  will show that all but at most  $o((n-t)!)$  permutations in  $\mathcal{A}$  fix some point of  $[t]$ , enabling us to complete the proof.

*Proof of Lemma 7:*

Let  $\mathcal{A} \subset S_n$  be a  $t$ -intersecting family of size at least  $c(n-t)!$ ; write  $\delta = 1 - c < 1$ . From (1), we know that the Euclidean distance from  $v_{\mathcal{A}}$  to  $V_t$  is small:

$$\|P_{V_t^\perp}(v_{\mathcal{A}})\|^2 \leq \delta(1 + O(n^{1/6}))|\mathcal{A}|/n!$$

From (4), the projection of  $v_{\mathcal{A}}$  onto  $V_t$  has  $\sigma$ -coordinate:

$$P_{V_t}(v_{\mathcal{A}})_\sigma = \frac{1}{n!} \sum_{\text{fat } \alpha} f^\alpha \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi \sigma^{-1})$$

Write  $P_\sigma = P_{V_t}(v_{\mathcal{A}})_\sigma$ ; then

$$\frac{1}{n!} \left( \sum_{\sigma \in \mathcal{A}} (1 - P_\sigma)^2 + \sum_{\sigma \notin \mathcal{A}} P_\sigma^2 \right) \leq \delta(1 + O(1/n^{1/6}))|\mathcal{A}|/n!$$

i.e.

$$\sum_{\sigma \in \mathcal{A}} (1 - P_\sigma)^2 + \sum_{\sigma \notin \mathcal{A}} P_\sigma^2 \leq \delta(1 + O(1/n^{1/6}))|\mathcal{A}|$$

Choose  $C > 0 : |\mathcal{A}|(1 - 1/n)\delta(1 + C/n^{1/6}) \geq \text{RHS}$ ; then the subset

$$\mathcal{S} := \{\sigma \in \mathcal{A} : (1 - P_\sigma)^2 < \delta(1 + C/n^{1/6})\}$$

has size at least  $|\mathcal{A}|/n$ . Similarly,  $P_\sigma^2 < 2\delta/n$  for all but at most

$$n|\mathcal{A}|(1 + O(1/n))/2$$

permutations  $\sigma \notin \mathcal{A}$ . Provided  $n$  is sufficiently large,  $|\mathcal{A}| \leq (n - t)!$ , and therefore the subset  $\mathcal{T} = \{\sigma \notin \mathcal{A} : P_\sigma^2 < 2\delta/n\}$  has size

$$|\mathcal{T}| \geq n! - (n - t)! - n(n - t)!(1 + O(1/n))/2$$

The permutations  $\sigma \in \mathcal{S}$  have  $P_\sigma$  close to 1; the permutations  $\pi \in \mathcal{T}$  have  $P_\pi$  close to 0. Using only our lower bounds on the sizes of  $\mathcal{S}$  and  $\mathcal{T}$ , we may prove the following:

*Claim:* There exist permutations  $\sigma \in \mathcal{S}$ ,  $\pi \in \mathcal{T}$  such that  $\sigma^{-1}\pi$  is a product of at most  $h = h(n)$  transpositions, where  $h = \sqrt{2(t + 2)(n - 1) \log n}$ .

*Proof of Claim:* Define the *transposition graph*  $H$  to be the Cayley graph on  $S_n$  generated by the transpositions, i.e.  $V(H) = S_n$  and  $\sigma\pi \in E(H)$  iff  $\sigma^{-1}\pi$  is a transposition. We use the following isoperimetric inequality for  $H$ , essentially the martingale inequality of Maurey:

**Theorem 8.** *Let  $X \subset V(H)$  with  $|X| \geq \gamma n!$  where  $0 < \gamma < 1$ . Then for any  $h \geq h_0 := \sqrt{\frac{1}{2}(n - 1) \log \frac{1}{\gamma}}$ ,*

$$|N_h(X)| \geq \left(1 - e^{-\frac{2(h-h_0)^2}{n-1}}\right) n!$$

□

For a proof, see for example [10]. Applying this to the set  $\mathcal{S}$ , which has  $|\mathcal{S}| \geq (1 - \delta)(n - t)!/n \geq \frac{n!}{n^{t+2}}$  (provided  $n$  is sufficiently large), with  $\gamma = 1/n^{t+2}$ ,  $h = 2h_0$ , gives  $|N_h(\mathcal{S})| \geq (1 - n^{-(t+2)})n!$ , so certainly  $N_h(\mathcal{S}) \cap \mathcal{T} \neq \emptyset$ , proving the claim.

We now have two permutations  $\sigma \in \mathcal{A}$ ,  $\pi \notin \mathcal{A}$  which are ‘close’ to one another in  $H$  (differing in only  $O(\sqrt{n \log n})$  transpositions) such that

$$P_\sigma > 1 - \sqrt{\delta(1 + C/n^{1/6})}, \quad P_\pi < \sqrt{2\delta/n}$$

and therefore

$$P_\sigma - P_\pi > 1 - \sqrt{\delta} - O(1/n^{1/12})$$

Hence, by averaging, there exist two permutations  $\rho, \tau$  that differ by just one transposition and satisfy

$$P_\rho - P_\tau > (1 - \sqrt{\delta} - O(1/n^{1/12}))/h \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t+2)n \log n}}$$

i.e.

$$\sum_{\alpha \in \mathcal{F}_{n,t}} \frac{f^\alpha}{n!} \left( \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi \rho^{-1}) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi \tau^{-1}) \right) \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t+2)n \log n}}$$

By double translation, we may assume without loss of generality that  $\rho = \text{Id}$ ,  $\tau = (1 \ 2)$ . So we have:

$$\sum_{\alpha \in \mathcal{F}_{n,t}} \frac{f^\alpha}{n!} \left( \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi(1 \ 2)) \right) \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t+2)n \log n}}$$

The above sum is over  $|\mathcal{F}_{n,t}| = \sum_{s=0}^t p(s)$  partitions  $\alpha$  of  $n$ ; this grows very rapidly with  $t$ , but is independent of  $n$  for  $n \geq 2t$ . By averaging, there exists some  $\alpha \in \mathcal{F}_{n,t}$  such that

$$\begin{aligned} \frac{f^\alpha}{n!} \left( \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi(1 \ 2)) \right) &\geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t+2)n \log n} \sum_{s=0}^t p(s)} \\ &= \Omega(1/\sqrt{n \log n}) \end{aligned}$$

Recall that the ‘fat’ irreducible representations  $\{[\alpha] : \alpha \in \mathcal{F}_{n,t}\}$  are precisely the irreducible constituents of  $M^{(n-t, 1^t)}$ , so very crudely, for each fat  $\alpha$ ,

$$f^\alpha \leq \dim(M^{(n-t, 1^t)}) = n(n-1) \dots (n-t+1)$$

Hence,

$$\sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi) - \sum_{\pi \in \mathcal{A}} \chi_\alpha(\pi(1 \ 2)) \geq \Omega(1/\sqrt{n \log n})(n-t)!$$

But for any  $\alpha \in \mathcal{F}_{n,t}$ , we may express the irreducible character  $\chi_\alpha$  as a linear combination of permutation characters  $\xi_\beta : \beta \in \mathcal{F}_{n,t}$  using the following ‘determinantal formula’ (see [8]). For any partition  $\alpha$  of  $n$ ,

$$\chi_\alpha = \sum_{\pi \in S_n} \epsilon(\pi) \xi_{\alpha - \text{id} + \pi}$$

Here, for  $\alpha = (\alpha_1, \dots, \alpha_l) \vdash n$ , we set  $\alpha_i = 0$  ( $l < i \leq n$ ), we think of  $\alpha$ ,  $\text{id}$  and  $\pi$  as sequences of length  $n$ , and we define addition and subtraction of these sequences pointwise. In general,

$$\alpha - \text{id} + \pi = (\alpha_1 - 1 + \pi(1), \alpha_2 - 2 + \pi(2), \dots, \alpha_n - n + \pi(n))$$

will be a sequence of  $n$  integers with sum  $n$ , i.e. a *composition* of  $n$ . If  $\lambda$  is a composition of  $n$  with all its terms non-negative, then let  $\bar{\lambda}$  be the partition of  $n$  produced by ordering the terms of  $\lambda$  in non-increasing order, and define  $\xi_\lambda = \xi_{\bar{\lambda}}$ ; if  $\lambda$  has a negative term, we define  $\xi_\lambda = 0$ . If  $\alpha \in \mathcal{F}_{n,t}$ , then as  $\alpha_1 \geq n - t$ , any composition occurring in the above sum has first term at least  $n - t$ , and therefore  $\xi_\beta$  can only occur in the above sum if  $\beta \in \mathcal{F}_{n,t}$ . Observe further that since  $\alpha$  has at most  $t + 1$  non-zero parts,  $\alpha_i = 0$  for every  $i > t + 1$ , and therefore any permutation  $\pi \in S_n$  with  $\xi_{\alpha - \text{id} + \pi} \neq 0$  must have  $\pi(i) \geq i$  for every  $i > t + 1$ , so must fix  $t + 2, t + 3, \dots$ , and  $n$ . Therefore, the above sum is only over  $\pi \in S_{\{1, \dots, t+1\}}$ , i.e.

$$\chi_\alpha = \sum_{\pi \in S_{t+1}} \epsilon(\pi) \xi_{\alpha - \text{id} + \pi} \quad \forall \alpha \in \mathcal{F}_{n,t}$$

Therefore,  $\chi_\alpha$  is a  $(\pm 1)$ -linear combination of at most  $(t + 1)!$  permutation characters  $\xi_\beta$  ( $\beta \in \mathcal{F}_{n,t}$ ), possibly with repeats. Hence, by averaging, there exists some  $\beta \in \mathcal{F}_{n,t}$  such that

$$\begin{aligned} \left| \sum_{\pi \in \mathcal{A}} \xi_\beta(\pi) - \sum_{\pi \in \mathcal{A}} \xi_\beta(\pi(1 \ 2)) \right| &\geq \Omega(1/\sqrt{n \log n}) \frac{(n - t)!}{(t + 1)!} \\ &= \Omega(1/\sqrt{n \log n}) (n - t)! \end{aligned}$$

Without loss of generality, we may assume that the above quantity is positive, i.e.

$$\sum_{\pi \in \mathcal{A}} \xi_\beta(\pi) - \sum_{\pi \in \mathcal{A}} \xi_\beta(\pi(1 \ 2)) \geq \Omega(1/\sqrt{n \log n}) (n - t)!$$

Let  $\mathbb{T}_\beta$  be the set of  $\beta$ -tabloids; the LHS is then

$$\begin{aligned} & \#\{(T, \pi) : T \in \mathbb{T}_\beta, \pi \in \mathcal{A}, \pi(T) = T\} \\ & - \#\{(T, \pi) : T \in \mathbb{T}_\beta, \pi \in \mathcal{A}, \pi(1\ 2)(T) = T\} \end{aligned}$$

Interchanging the order of summation, this equals

$$\sum_{T \in \mathbb{T}_\beta} (\#\{\pi \in \mathcal{A} : \pi(T) = T\} - \#\{\pi \in \mathcal{A} : \pi(1\ 2)(T) = T\})$$

The above summand is zero for all  $\beta$ -tabloids  $T$  with 1 and 2 in the first row of  $T$  (as then  $(1\ 2)T = T$ ). Write  $\beta = (n - s, \beta_2, \dots, \beta_l)$ , where  $0 \leq s \leq t$ . The number of  $\beta$ -tabloids with 1 not in the first row is

$$s(n-1)(n-2) \dots (n-s+1) / \prod_{i=2}^l \beta_i!$$

and therefore the number of  $\beta$ -tabloids with 1 or 2 below the first row is at most

$$\begin{aligned} 2s(n-1)(n-2) \dots (n-s+1) / \prod_{i=2}^l \beta_i! & \leq 2t(n-1)(n-2) \dots (n-s+1) \\ & = \frac{2t(n-1)!}{(n-s)!} \end{aligned}$$

Hence by averaging, for one such  $\beta$ -tabloid  $T$ ,

$$\begin{aligned} & \#\{\pi \in \mathcal{A} : \pi(T) = T\} - \#\{\pi \in \mathcal{A} : \pi(1\ 2)(T) = T\} \\ & \geq \Omega(1/\sqrt{n \log n}) \frac{(n-s)!}{2t(n-1)!} (n-t)! \end{aligned}$$

and therefore the number of permutations in  $\mathcal{A}$  fixing  $T$  satisfies

$$\#\{\pi \in \mathcal{A} : \pi(T) = T\} \geq \Omega(1/\sqrt{n \log n}) \frac{(n-s)!}{2t(n-1)!} (n-t)!$$

Without loss of generality, we may assume that the first row of  $T$  consists of the numbers  $\{s+1, \dots, n\}$ . There are  $\beta_2! \beta_3! \dots \beta_l! \leq s! \leq t!$  permutations of  $[s]$  fixing the 2<sup>nd</sup>, 3<sup>rd</sup>, ..., and  $l^{\text{th}}$  rows of  $T$ ; any permutation fixing  $T$  must agree with one of these permutations on  $[s]$ . Hence, there exists a permutation  $\rho$  of  $[s]$  such that at least

$$\Omega(1/\sqrt{n \log n}) \frac{(n-s)!(n-t)!}{2t(n-1)!t!}$$



permutations in  $\mathcal{A}$  agree with  $\rho$  on  $[s]$ . Without loss of generality, we may assume that  $\rho = \text{Id}_{[s]}$ , so the number of permutations in  $\mathcal{A}$  fixing  $[s]$  pointwise satisfies

$$\begin{aligned} |\mathcal{A}_{1 \mapsto 1, \dots, s \mapsto s}| &\geq \Omega(1/\sqrt{n \log n}) \frac{(n-s)!(n-t)!}{2t(n-1)!t!} \\ &= \Omega(1/\sqrt{n \log n}) \frac{(n-s)!(n-t)!}{(n-1)!} \end{aligned}$$

We may write  $\mathcal{A}_{1 \mapsto 1, \dots, s \mapsto s}$  as a disjoint union

$$\mathcal{A}_{1 \mapsto 1, \dots, s \mapsto s} = \bigcup_{j_{s+1}, \dots, j_t > s \text{ distinct}} \mathcal{A}_{1 \mapsto 1, \dots, s \mapsto s, s+1 \mapsto j_{s+1}, \dots, t \mapsto j_t}$$

and there are  $(n-s)(n-s-1) \dots (n-t+1)$  choices of  $j_{s+1}, \dots, j_t$ , so by averaging, there exists a choice such that

$$|\mathcal{A}_{1 \mapsto 1, \dots, s \mapsto s, s+1 \mapsto j_{s+1}, \dots, t \mapsto j_t}| \geq \Omega(1/\sqrt{n \log n}) \frac{((n-t)!)^2}{(n-1)!}$$

By translation, we may assume without loss of generality that  $j_k = k$  for each  $k$ , so

$$\begin{aligned} |\mathcal{A}_{1 \mapsto 1, 2 \mapsto 2, \dots, t \mapsto t}| &\geq \Omega(1/\sqrt{n \log n}) \frac{((n-t)!)^2}{(n-1)!} \\ &= \Omega(\sqrt{n/\log n})(n-2t)! \\ &= \omega((n-2t)!) \end{aligned}$$

We will use this to show that the number of permutations in  $\mathcal{A}$  with no fixed point in  $[t]$  is small. We may write

$$\mathcal{A} \setminus (\mathcal{A}_{1 \mapsto 1} \cup \dots \cup \mathcal{A}_{t \mapsto t}) = \bigcup_{j_1, \dots, j_t \text{ distinct} : j_k \neq k \ \forall k \in [t]} \mathcal{A}_{1 \mapsto j_1, \dots, t \mapsto j_t}$$

We now show that each  $\mathcal{A}_{1 \mapsto j_1, \dots, t \mapsto j_t}$  is small using Theorem 4. Let  $J = \{j_1, \dots, j_t\}$ . Notice that  $\mathcal{E} := \mathcal{A}_{1 \mapsto 1, \dots, t \mapsto t}$ ,  $\mathcal{F} := \mathcal{A}_{1 \mapsto j_1, \dots, t \mapsto j_t}$  is a  $t$ -cross-intersecting pair of families, so for any  $\sigma \in \mathcal{E}$  and  $\pi \in \mathcal{F}$ , there are  $t$  distinct points  $i_1, i_2, \dots, i_t > t$  such that  $\sigma(i_k) = \pi(i_k) \notin [t] \cup J$  for each  $k \in [t]$ . But then

$$(1 \ j_1)(2 \ j_2) \dots (t \ j_t) \pi(i_k) = \sigma(i_k) \quad \text{for each } k \in [t]$$

so letting  $\mathcal{G} := (1\ j_1)(2\ j_2) \dots (t\ j_t)\mathcal{F}$ , the pair of families  $\mathcal{E}, \mathcal{G}$  fix  $[t]$  pointwise and  $t$ -cross-intersect on  $\{t+1, t+2, \dots, n\}$ . Deleting  $1, \dots, t$  we obtain a  $t$ -cross-intersecting pair  $\mathcal{E}', \mathcal{G}'$  of subsets of  $S_{\{t+1, \dots, n\}}$ . By Theorem 4,

$$|\mathcal{A}_{1 \mapsto 1, \dots, t \mapsto t}| |\mathcal{A}_{1 \mapsto j_1, \dots, t \mapsto j_t}| = |\mathcal{E}| |\mathcal{G}| = |\mathcal{E}'| |\mathcal{G}'| \leq ((n-2t)!)^2$$

Since

$$|\mathcal{A}_{1 \mapsto 1, \dots, t \mapsto t}| \geq \omega((n-2t)!)$$

we have

$$|\mathcal{A}_{1 \mapsto j_1, \dots, t \mapsto j_t}| \leq o((n-2t)!)$$

There are  $\leq n(n-1)(n-2) \dots (n-t+1)$  possible choices of  $j_1, \dots, j_t$ , and therefore the number of permutations in  $\mathcal{A}$  with no fixed point in  $[t]$  satisfies

$$\begin{aligned} |\mathcal{A} \setminus (\mathcal{A}_{1 \mapsto 1} \cup \mathcal{A}_{2 \mapsto 2} \cup \dots \cup \mathcal{A}_{t \mapsto t})| &\leq o((n-2t)!)n(n-1) \dots (n-t+1) \\ &= o((n-t)!) \end{aligned}$$

Since  $|\mathcal{A}| \geq c(n-t)!$ , we have

$$|\mathcal{A}_{1 \mapsto 1} \cup \mathcal{A}_{2 \mapsto 2} \cup \dots \cup \mathcal{A}_{t \mapsto t}| \geq (c - o(1))(n-t)!$$

By averaging, there exists some  $i \in [t]$  such that

$$|\mathcal{A}_{i \mapsto i}| \geq (c - o(1))(n-t)!/t$$

We may assume that  $i = 1$ , so  $|\mathcal{A}_{1 \mapsto 1}| \geq (c - o(1))(n-t)!/t$ . Now, using the same trick as before, we may use Theorem 4 to show that  $|\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}| \leq O((n-t-1)!)$ . Indeed, write  $\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}$  as a disjoint union

$$\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1} = \bigcup_{j \neq 1} \mathcal{A}_{1 \mapsto j}$$

We will show that each  $\mathcal{A}_{1 \mapsto j}$  is small. Notice as before that the pair of families  $\mathcal{A}_{1 \mapsto 1}, (1\ j)\mathcal{A}_{1 \mapsto j}$  fixes 1 and  $t$ -cross-intersects on the domain  $\{2, \dots, n\}$ , so Theorem 4 gives

$$|\mathcal{A}_{1 \mapsto 1}| |\mathcal{A}_{1 \mapsto j}| \leq ((n-t-1)!)^2$$

Since  $|\mathcal{A}_{1 \mapsto 1}| \geq \Omega((n-t)!)$ , we obtain  $|\mathcal{A}_{1 \mapsto j}| \leq O((n-t-2)!)$ , and therefore

$$|\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}| = \sum_{j \neq 1} |\mathcal{A}_{1 \mapsto j}| \leq O((n-t-1)!)$$

proving Lemma 7.

□

*Proof of Theorem 6:*

By induction on  $t$ . The  $t = 1$  case is the same as that of Lemma 7. Assume the theorem is true for  $t - 1$ ; we will prove it for  $t$ . Let  $\mathcal{A} \subset S_n$  be a  $t$ -intersecting family of size at least  $c(n - t)!$ . By Lemma 7, there exist  $i$  and  $j$  such that  $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n - t - 1)!)$ . Without loss of generality we may assume that  $i = j = 1$ , so  $|\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}| \leq O((n - t - 1)!)$ . Hence,  $|\mathcal{A}_{1 \mapsto 1}| \geq |\mathcal{A}| - O((n - t - 1)!)$ . Deleting 1 from each permutation in  $\mathcal{A}_{1 \mapsto 1}$ , we obtain a  $(t - 1)$ -intersecting family  $\mathcal{A}' \subset S_{\{2, 3, \dots, n\}}$  of size  $\geq (c - O(1/n))(n - t)!$ . Choose any positive constant  $c' < c$ ; then provided  $n$  is sufficiently large, we have  $|\mathcal{A}'| \geq c'(n - t)!$ . By the induction hypothesis, there exists a  $(t - 1)$ -coset  $\mathcal{C}'$  of  $S_{2, \dots, n}$  such that  $|\mathcal{A}' \setminus \mathcal{C}'| \leq O((n - t - 1)!)$ . Then if  $\mathcal{C}$  is the  $t$ -coset obtained from  $\mathcal{C}'$  by adjoining  $1 \mapsto 1$ , we have  $|\mathcal{A} \setminus \mathcal{C}| \leq O((n - t - 1)!)$ . This completes the induction and proves Theorem 6.

□

We now use our rough stability result to prove an exact stability result. First, we need some more definitions.

Let  $d_n$  be the number of *derangements* of  $[n]$  (permutations of  $[n]$  without fixed points). It is well known that  $d_n = (1/e + o(1))n!$ .

Following Cameron and Ku [3], given a permutation  $\rho \in S_n$  and  $i \in [n]$ , we define the *i-fix* of  $\rho$  to be the permutation  $\rho_i$  which fixes  $i$ , maps the preimage of  $i$  to the image of  $i$ , and agrees with  $\rho$  at all other points of  $[n]$ , i.e.

$$\rho_i(i) = i; \quad \rho_i(\rho^{-1}(i)) = \rho(i); \quad \rho_i(k) = \rho(k) \quad \forall k \neq i, \rho^{-1}(i)$$

In other words,  $\rho_i = \rho(\rho^{-1}(i) \ i)$ . We inductively define

$$\rho_{i_1, \dots, i_l} = (\rho_{i_1, \dots, i_{l-1}})_{i_l}$$

Notice that if  $\sigma$  fixes  $j$ , then  $\sigma$  agrees with  $\rho_j$  wherever it agrees with  $\rho$ .

**Theorem 9.** *For  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family which is not contained within a  $t$ -coset, then  $\mathcal{A}$  is no larger than the family*

$$\begin{aligned} \mathcal{D} = & \{ \sigma \in S_n : \sigma(i) = i \ \forall i \leq t, \sigma(j) = j \text{ for some } j > t + 1 \} \\ & \cup \{ (1 \ t + 1), (2 \ t + 1), \dots, (t \ t + 1) \} \end{aligned}$$

which has size  $(n - t)! - d_{n-t} - d_{n-t-1} + t = (1 - 1/e + o(1))(n - t)!$ . If  $\mathcal{A}$  is the same size as  $\mathcal{D}$ , then  $\mathcal{A}$  is a double translate of  $\mathcal{D}$ , i.e.  $\mathcal{A} = \pi \mathcal{D} \tau$  for some  $\pi, \tau \in S_n$ .

*Proof.* Suppose  $\mathcal{A} \subset S_n$  is a  $t$ -intersecting family which is not contained within a  $t$ -coset, and has size

$$|\mathcal{A}| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!.$$

Applying Theorem 6 with any constant  $c$  such that  $0 < c < 1 - 1/e$ , we see that (provided  $n$  is sufficiently large) there exists a  $t$ -coset  $\mathcal{C}$  such that

$$|\mathcal{A} \setminus \mathcal{C}| \leq O(1/n)(n-t)!$$

By double translation, without loss of generality we may assume that  $\mathcal{C} = \{\sigma \in S_n : \sigma(1) = 1, \dots, \sigma(t) = t\}$ . We have:

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\geq (n-t)! - d_{n-t} - d_{n-t-1} + t - O(1/n)(n-t)! \\ &= (1 - 1/e + o(1))(n-t)! \end{aligned} \tag{5}$$

We now claim that every permutation in  $\mathcal{A} \setminus \mathcal{C}$  fixes exactly  $t-1$  points of  $[t]$ . Suppose for a contradiction that  $\mathcal{A}$  contains a permutation  $\tau$  fixing at most  $t-2$  points of  $[t]$ . Then every permutation in  $\mathcal{A} \cap \mathcal{C}$  must agree with  $\tau$  on at least 2 points of  $\{t+1, \dots, n\}$ , so

$$|\mathcal{A} \cap \mathcal{C}| \leq \binom{n-t}{2} (n-t-2)! = \frac{1}{2}(n-t)!$$

contradicting (5), provided  $n$  is sufficiently large.

Since we are assuming that  $\mathcal{A}$  is not contained within a  $t$ -coset,  $\mathcal{A} \setminus \mathcal{C}$  contains some permutation  $\tau$ ;  $\tau$  must fix all points of  $[t]$  except for one. By double translation, we may assume that  $\tau = (1 \ t+1)$ . We will show that under these hypotheses,  $\mathcal{A} = \mathcal{D}$ .

Every permutation in  $\mathcal{A} \cap \mathcal{C}$  must  $t$ -intersect  $(1 \ t+1)$  and must therefore have at least one fixed point  $> t+1$ , i.e.  $\mathcal{A} \cap \mathcal{C}$  is a subset of the family

$$\mathcal{E} := \{\sigma \in S_n : \sigma(i) = i \ \forall i \in [t], \sigma(j) = j \text{ for some } j > t+1\}$$

which has size

$$(n-t)! - d_{n-t} - d_{n-t-1}$$

We now make the following observation:

*Claim:*  $\mathcal{A} \setminus \mathcal{C}$  may only contain the transpositions  $\{(i \ t+1) : i \in [t]\}$ .

*Proof of Claim:*

Suppose for a contradiction that  $\mathcal{A} \setminus \mathcal{C}$  contains a permutation  $\rho$  not of this

form. Then  $\rho(j) \neq j$  for some  $j \geq t+2$ . We will show that there are at least  $d_{n-t-1}$  permutations in  $\mathcal{E}$  which fix  $j$  and disagree with  $\rho$  at every point of  $\{t+1, t+2, \dots, n\}$ , and therefore cannot  $t$ -intersect  $\rho$ . Let  $l$  be the unique point of  $[t]$  not fixed by  $\rho$ . If  $\sigma$  fixes both  $l$  and  $j$ , then  $\sigma$  agrees with  $\rho_{j,l} = (\rho_j)_l$  wherever it agrees with  $\rho$ . Notice that  $\rho_{j,l}$  fixes  $1, 2, \dots, t$  and  $j$ . There are exactly  $d_{n-t-1}$  permutations in  $\mathcal{E}$  which fix  $j$  and disagree with  $\rho_{j,l}$  at every point of  $\{t+1, t+2, \dots, n\} \setminus \{j\}$ ; each disagrees with  $\rho$  at every point of  $\{t+1, t+2, \dots, n\}$ . So none  $t$ -intersect  $\rho$ , so none are in  $\mathcal{A}$ , and therefore

$$|\mathcal{A} \cap \mathcal{C}| \leq |\mathcal{E}| - d_{n-t-1} = (n-t)! - d_{n-t} - 2d_{n-t-1}$$

Since we are assuming that  $|\mathcal{A}| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t$ , this means that

$$|\mathcal{A} \setminus \mathcal{C}| \geq d_{n-t-1} + t = (1/e + o(1))(n-t-1)!$$

Notice that for any  $m \leq n$  we have the following trivial upper bound on the size of an  $m$ -intersecting family  $\mathcal{H} \subset S_n$ :

$$|\mathcal{H}| \leq \binom{n}{m} (n-m)! = n!/m!$$

since every permutation in  $\mathcal{H}$  must agree with a fixed permutation in  $\mathcal{H}$  in at least  $m$  places.

Hence,  $\mathcal{A} \setminus \mathcal{C}$  cannot be  $(\log n)$ -intersecting and therefore contains two permutations  $\pi, \tau$  agreeing on at most  $\log n$  points. The number of permutations fixing  $[t]$  pointwise and agreeing with both  $\pi$  and  $\tau$  at one of these  $\log n$  points is therefore at most  $(\log n)(n-t-1)!$ . All other permutations in  $\mathcal{A} \cap \mathcal{C}$  agree with  $\pi$  and  $\tau$  at two separate points of  $\{t+1, \dots, n\}$ , and by the above argument, the same holds for  $\pi_p$  and  $\tau_q$ , where  $p$  and  $q$  are the points of  $[t]$  shifted by  $\pi$  and  $\tau$  respectively. The number of permutations in  $\mathcal{C}$  that agree with  $\pi_p$  and  $\tau_q$  at two separate points of  $\{t+1, \dots, n\}$  is at most  $((1-1/e)^2 + o(1))(n-t)!$  (it is easily checked that given two fixed permutations, the probability that a uniform random permutation agrees with them at separate points is at most  $(1-1/e)^2 + o(1)$ ), which implies that

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\leq ((1-1/e)^2 + o(1))(n-t)! + (\log n)(n-t-1)! \\ &= ((1-1/e)^2 + o(1))(n-t)! \end{aligned}$$

contradicting (5), provided  $n$  is sufficiently large. This proves the claim.

Since we are assuming  $|\mathcal{A}| \geq |\mathcal{E}| + t$ , we must have equality, so  $\mathcal{A} = \mathcal{D}$ , proving Theorem 9.  $\square$

Similar arguments give the following stability results for  $t$ -cross-intersecting families. Say two pairs of families  $(\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D})$  in  $S_n$  are *isomorphic* if there exist permutations  $\pi, \rho \in S_n$  such that  $\mathcal{A} = \pi\mathcal{C}\rho$  and  $\mathcal{B} = \pi\mathcal{D}\rho$ . We have:

**Theorem 10.** *For  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A}, \mathcal{B} \subset S_n$  are  $t$ -cross-intersecting but not both contained within the same  $t$ -coset, then*

$$\min(|\mathcal{A}|, |\mathcal{B}|) \leq (n-t)! - d_{n-t} - d_{n-t-1} + t$$

*with equality iff  $(\mathcal{A}, \mathcal{B})$  is isomorphic to the pair of families*

$$\begin{aligned} & \{\sigma : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = \tau(j) \text{ for some } j > t+1\} \cup \{(i \ t+1) : i \in [t]\} \\ & \{\sigma : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \text{ for some } j > t+1\} \cup \{(1i)\tau(1i) : i \in [t]\} \end{aligned}$$

*where  $\tau(1) \neq 1$  and if  $t \geq 2$ ,  $\tau$  fixes  $2, 3, \dots, t$  and at least two points  $> t+1$ , whereas if  $t = 1$ ,  $\tau$  intersects  $(1 \ 2)$ .*

**Theorem 11.** *For  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A}, \mathcal{B} \subset S_n$  are  $t$ -cross-intersecting but not both contained within the same  $t$ -coset, then*

$$|\mathcal{A}||\mathcal{B}| \leq ((n-t)! - d_{n-t} - d_{n-t-1})((n-t)! + t)$$

*with equality iff  $(\mathcal{A}, \mathcal{B})$  is isomorphic to the pair of families*

$$\begin{aligned} & \{\sigma \in S_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \text{ for some } j > t+1\} \\ & \{\sigma \in S_n : \sigma(i) = i \ \forall i \leq t\} \cup \{(1 \ t+1), (2 \ t+1), \dots, (t \ t+1)\} \end{aligned}$$

The proofs are very similar to the proof of Theorem 9, and we omit them.

## 4 The Alternating Group

We now turn our attention to the alternating group  $A_n$ , the index-2 subgroup of  $S_n$  consisting of the even permutations of  $\{1, 2, \dots, n\}$ . The following may be deduced from the proof of the Deza-Frankl conjecture in [6]:

**Theorem 12.** *For  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A} \subset A_n$  is  $t$ -intersecting, then  $|\mathcal{A}| \leq (n-t)!/2$ .*

*Remark:* This implies the Deza-Frankl conjecture. To see this, let  $\mathcal{A} \subset S_n$  be  $t$ -intersecting; then  $\mathcal{A} \cap A_n$  and  $(\mathcal{A} \setminus A_n)(1 \ 2)$  are both  $t$ -intersecting families of permutations in  $A_n$ , so by Theorem 12, both have size at most  $(n-t)!/2$ . Hence,

$$|\mathcal{A}| = |\mathcal{A} \cap A_n| + |\mathcal{A} \setminus A_n| \leq (n-t)!$$

*Proof.* Recall that in [6], we constructed a weighted graph  $Y_{\text{even}}$  which was a real linear combination of Cayley graphs on  $S_n$  generated by conjugacy-classes of *even* permutations with less than  $t$  fixed points, and whose matrix of weights had maximum eigenvalue 1 and minimum eigenvalue

$$\omega_{n,t} = -\frac{1}{n(n-1)\dots(n-t+1)-1}$$

Clearly,  $Y_{\text{even}}$  has no (non-zero) edges between  $A_n$  and  $S_n \setminus A_n$ . Let  $Y_1$  be the weighted subgraph of  $Y_{\text{even}}$  induced on  $A_n$ , and  $Y_2$  the weighted subgraph induced on  $S_n \setminus A_n$ . Notice that the map

$$\begin{aligned} \phi : A_n &\rightarrow S_n \setminus A_n; \\ \sigma &\mapsto (1\ 2)\sigma \end{aligned}$$

is a graph isomorphism from  $Y_1$  to  $Y_2$ . To see this, note that

$$\phi(\sigma)(\phi(\pi))^{-1} = ((1\ 2)\sigma)((1\ 2)\pi)^{-1} = (1\ 2)\sigma\pi^{-1}(1\ 2)$$

which is conjugate to  $\sigma\pi^{-1}$ . Since  $Y_{\text{even}}$  is a linear combination of Cayley graphs generated by conjugacy-classes of  $S_n$ , the edge  $\phi(\sigma)\phi(\pi)$  has the same weight in  $Y_{\text{even}}$  as the edge  $\sigma\pi$ . Hence,  $Y_{\text{even}}$  is a disjoint union of the two isomorphic subgraphs  $Y_1$  and  $Y_2$ , so the eigenvalues of  $Y_{\text{even}}$  are the same as those of  $Y_1$  (with double the multiplicities). Applying Theorem 1 to  $Y_1$  proves Theorem 12.  $\square$

Our next aim is to show that equality holds in Theorem 12 only if  $\mathcal{A}$  is a coset of the stabilizer of  $t$  points. As for  $S_n$ , we will call these families the ‘ $t$ -cosets of  $A_n$ ’.

Let  $W_t$  be the subspace of  $\mathbb{C}[A_n]$  spanned by the characteristic vectors of the  $t$ -cosets of  $A_n$ . It is easily checked that  $W_t$  is the direct sum of the 1 and  $\omega_{n,t}$ -eigenspaces of  $Y_1$ . Hence, by Theorem 1, if equality holds in Theorem 12, then the characteristic vector  $v_{\mathcal{A}}$  of  $\mathcal{A}$  lies in the subspace  $W_t$ .

We would like to show that the Boolean functions which are linear combinations of the characteristic functions of the  $t$ -cosets of  $A_n$  are precisely the characteristic functions of the disjoint unions of  $t$ -cosets of  $A_n$ . To do this for  $S_n$  in [6], it was first proved that if a non-negative function  $f : S_n \rightarrow \mathbb{R}_{\geq 0}$  is a linear combination of the characteristic functions of the  $t$ -cosets of  $S_n$ , then it can be expressed as a linear combination of them with non-negative coefficients. However, this is not true in the case of  $A_n$ , even for  $t = 1$ :

*Claim:* There exists a non-negative function in  $W_1$  which cannot be written as a non-negative linear combination of the characteristic functions of the 1-cosets of  $A_n$ .

*Proof of Claim:* Let  $w_{i \rightarrow j}$  be the characteristic function of the 1-coset  $\{\sigma \in A_n : \sigma(i) = j\}$ . We say a real  $n \times n$  matrix  $B$  represents a function  $f \in W_1$  if  $f$  can be written as a linear combination of  $w_{i \rightarrow j}$ 's with coefficients given by the matrix  $B$ , i.e.

$$f = \sum_{i,j=1}^n b_{i,j} w_{i \rightarrow j}$$

or equivalently,

$$f(\sigma) = \sum_{i=1}^n b_{i,\sigma(i)} \quad \forall \sigma \in A_n$$

It is easy to see that, provided  $n \geq 4$ , any function  $f \in W_1$  has a unique extension to a function  $\tilde{f} \in V_1$ . Hence, if  $B$  and  $C$  are two matrices both representing  $f$ , they must both represent the same function  $\tilde{f} : S_n \rightarrow \mathbb{R}$ , and therefore

$$\sum_{i=1}^n b_{i,\sigma(i)} = \sum_{i=1}^n c_{i,\sigma(i)} \quad \forall \sigma \in S_n$$

Now let  $f$  be the function represented by the matrix

$$B = \begin{pmatrix} 1 & -1/2 & 1 & 1 & \dots & 1 \\ -1/2 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & 1 & \dots & & & 0 \end{pmatrix}$$

This takes only non-negative values on  $A_n$ , since

$$\sum_{i=1}^n b_{i,\sigma(i)} \geq 0 \quad \forall \sigma \in A_n$$

but if  $\tau$  is the transposition  $(1 \ 2)$ , then

$$\sum_{i=1}^n b_{i,\tau(i)} = -1$$



Hence, any matrix  $C$  representing the same function as  $B$  must also have

$$\sum_{i=1}^n c_{i,\tau(i)} = -1$$

and therefore cannot have non-negative entries. Therefore,  $f$  is a non-negative function in  $W_1$  that cannot be written as a non-negative linear combination of the  $w_{i \rightarrow j}$ 's, proving the claim.

Instead, we obtain our desired characterization of equality in Theorem 12 from a stability result for  $t$ -intersecting families in  $A_n$ .

Let  $e_n, o_n$  denote the number of respectively even/odd derangements of  $[n]$ . It is well known that  $e_n - o_n = (-1)^{n-1}(n-1) \forall n \in \mathbb{N}$ ; combining this with the fact that  $d_n = (1/e + o(1))n!$  gives  $e_n = (1/(2e) + o(1))n!$ ,  $o_n = (1/(2e) + o(1))n!$ .

We now prove the following analogue of Theorem 9:

**Theorem 13.** *For  $n$  sufficiently large depending on  $t$ , if  $\mathcal{A} \subset A_n$  is a  $t$ -intersecting family which is not contained within a  $t$ -coset of  $A_n$ , then  $\mathcal{A}$  cannot be larger than the family*

$$\begin{aligned} \mathcal{B} = & \{ \sigma \in A_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = (n-1 \ n)(j) \text{ for some } j > t+1 \} \\ & \cup \{ (1 \ t+1)(n-1 \ n), (2 \ t+1)(n-1 \ n), \dots, (t \ t+1)(n-1 \ n) \} \end{aligned}$$

which has size  $(n-t)!/2 - o_{n-t} - o_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!/2$ . If  $\mathcal{A}$  is the same size as  $\mathcal{B}$ , then  $\mathcal{A}$  is a double translate of  $\mathcal{B}$ , meaning that  $\mathcal{A} = \pi\mathcal{B}\tau$  for some  $\pi, \tau \in A_n$ .

*Proof.* Let  $\mathcal{A} \subset A_n$  be a  $t$ -intersecting family which is not contained within a  $t$ -coset of  $A_n$  and has size

$$|\mathcal{A}| \geq (n-t)!/2 - o_{n-t} - o_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!/2.$$

Applying Theorem 6 with any constant  $c$  such that  $0 < c < (1 - 1/e)/2$ , we see that (provided  $n$  is sufficiently large) there exists a  $t$ -coset  $\mathcal{C}$  such that

$$|\mathcal{A} \setminus \mathcal{C}| \leq O(1/n)(n-t)!$$

By double translation, without loss of generality we may assume that  $\mathcal{C} = \{ \sigma \in A_n : \sigma(1) = 1, \dots, \sigma(t) = t \}$ . We have:

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| & \geq (n-t)!/2 - o_{n-t} - o_{n-t-1} + t - O(1/n)(n-t)! \\ & = (1 - 1/e + o(1))(n-t)!/2 \end{aligned} \tag{6}$$

We now claim that every permutation in  $\mathcal{A} \setminus \mathcal{C}$  fixes exactly  $t - 1$  points of  $[t]$ . Suppose for a contradiction that  $\mathcal{A}$  contains a permutation  $\tau$  fixing at most  $t - 2$  points of  $[t]$ . Then every permutation in  $\mathcal{A} \cap \mathcal{C}$  must agree with  $\tau$  on at least 2 points of  $\{t + 1, \dots, n\}$ , so

$$|\mathcal{A} \cap \mathcal{C}| \leq \binom{n-t}{2} (n-t-2)!/2 = \frac{1}{2}(n-t)!/2$$

contradicting (6), provided  $n$  is sufficiently large.

Since we are assuming that  $\mathcal{A}$  is not contained within a  $t$ -coset,  $\mathcal{A} \setminus \mathcal{C}$  contains some permutation  $\tau$ ;  $\tau$  must fix all points of  $[t]$  except for one. By double translation, we may assume that  $\tau = (1 \ t+1)(n-1 \ n)$ . We will show that under these hypotheses,  $\mathcal{A} = \mathcal{B}$ . Every permutation in  $\mathcal{A} \cap \mathcal{C}$  must agree with  $(n-1 \ n)$  at some point  $\geq t+2$ , i.e.  $\mathcal{A} \cap \mathcal{C}$  is a subset of the family

$$\mathcal{E} := \{\sigma \in A_n : \sigma(i) = i \ \forall i \in [t], \ \sigma(j) = (n-1 \ n)(j) \text{ for some } j \geq t+2\}$$

which has size

$$(n-t)!/2 - o_{n-t} - o_{n-t-1}$$

We now make the following observation:

*Claim:*  $\mathcal{A} \setminus \mathcal{C}$  may only contain the permutations  $\{(i \ t+1)(n-1 \ n) : i \in [t]\}$ .

*Proof of Claim:*

Suppose for a contradiction that  $\mathcal{A} \setminus \mathcal{C}$  contains a permutation  $\rho$  not of this form. Then  $\rho(j) \neq (n-1 \ n)(j)$  for some  $j \geq t+2$ , so by a very similar argument to in the proof of Theorem 9, there are at least  $\min(e_{n-t-1}, o_{n-t-1})$  even permutations which fix  $1, 2, \dots, t$  and agree with  $(n-1 \ n)$  at  $j$  (and are therefore in  $\mathcal{E}$ ) and also disagree with  $\rho$  at all points of  $\{t+1, t+2, \dots, n\} \setminus \{j\}$ . Since  $\rho$  has exactly  $t-1$  fixed points in  $[t]$ , none of these permutations can  $t$ -intersect  $\rho$ , and therefore

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\leq |\mathcal{E}| - \min(e_{n-t-1}, o_{n-t-1}) \\ &= (n-t)! - o_{n-t} - o_{n-t-1} - \min(e_{n-t-1}, o_{n-t-1}) \end{aligned}$$

Since we are assuming that  $|\mathcal{A}| \geq (n-t)! - o_{n-t} - o_{n-t-1} + t$ , this means that

$$|\mathcal{A} \setminus \mathcal{C}| \geq \min(e_{n-t-1}, o_{n-t-1}) + t = (1/e + o(1))(n-t-1)!/2$$

Notice that for any  $m < n$  we have the following trivial upper bound on the size of an  $m$ -intersecting family  $\mathcal{H} \subset A_n$ :

$$|\mathcal{H}| \leq \binom{n}{m} (n-m)!/2 = n!/(2m!)$$

since every permutation in  $\mathcal{H}$  must agree with a fixed permutation in  $\mathcal{H}$  in at least  $m$  places.

Hence,  $\mathcal{A} \setminus \mathcal{C}$  cannot be  $(\log n)$ -intersecting and therefore contains two permutations  $\pi, \tau$  agreeing on at most  $\log n$  points. The number of permutations in  $\mathcal{C}$  which agree with  $\pi$  and  $\tau$  at one of these  $\log n$  points is clearly at most  $(\log n)(n-t-1)!/2$ . All other permutations in  $\mathcal{A} \cap \mathcal{C}$  agree with  $\pi$  and  $\tau$  at two separate points of  $\{t+1, \dots, n\}$ , and therefore the same holds for  $\pi_p$  and  $\tau_q$ , where  $p$  and  $q$  are the unique points of  $[t]$  shifted by  $\pi$  and  $\tau$  respectively. The number of permutations in  $\mathcal{C}$  that agree with  $\pi_p$  and  $\tau_q$  at two separate points of  $\{t+1, \dots, n\}$  is at most  $((1-1/e)^2 + o(1))(n-t)!/2$  (it is easily checked that given two fixed permutations, the probability that a uniform random even permutation agrees with them at separate points is at most  $(1-1/e)^2 + o(1)$ ), which implies that

$$\begin{aligned} |\mathcal{A} \cap \mathcal{C}| &\leq ((1-1/e)^2 + o(1))(n-t)!/2 + (\log n)(n-t-1)!/2 \\ &= ((1-1/e)^2 + o(1))(n-t)!/2 \end{aligned}$$

contradicting (6), provided  $n$  is sufficiently large. This proves the claim.

Since we are assuming  $|\mathcal{A}| \geq |\mathcal{E}| + t$ , we must have equality, so  $\mathcal{A} = \mathcal{B}$ , proving Theorem 13.  $\square$

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